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ON $\Psi_{\alpha,\beta}$–EXPANSIVE MAPPINGS WITH DISPLACEMENT CONTROL AND FIXED POINT CONSEQUENCES

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Abstract: In this paper, we introduce a new class of nonlinear expansive mappings governed by a rational displacement–distance gauge $\Psi_{\alpha,\beta}$, which simultaneously depends on the interpoint distance and the individual self–displacements of the operator. This framework extends classical Wang–type expansive models that are based solely on interpoint distances. Under a natural domination condition linking displacement and distance, we establish the existence, uniqueness, and global convergence of fixed points for $\Psi_{\alpha,\beta}$–expansive mappings in complete metric spaces. The proposed approach yields a displacement–sensitive expansive mechanism that enables the treatment of operators not covered by classical expansive conditions, thereby overcoming limitations of existing theories and providing a more flexible framework for applications in nonlinear analysis. Several nontrivial examples are presented to illustrate the applicability, strength, and novelty of the proposed theory.

Keywords and Phrases: Expansive mappings; fixed points; displacement control; rational gauge; backward iteration; metric spaces.

2020 Mathematics Subject Classification: 47H10, 54H25, 54E50.

1. Introduction

Since the pioneering work of Banach [5], fixed point theory has evolved into a fundamental framework in nonlinear analysis with wide-ranging applications in both pure and applied mathematics. A variety of contractive conditions have been developed in complete metric spaces, including nonlinear contractions due to Boyd and Wong [6], Meir and Keeler [21], and classical mappings of Kannan [19] and Chatterjea [8]. Further important generalizations were introduced by Geraghty [13] and Čirić [9], as well as approaches based on altering distance functions by Khan, Swalech and Sessa [20]. These ideas have been extended to various generalized metric structures, such as b–metric spaces by Czerwik [10], generalized metric spaces by Branciari [7], partial metric spaces by Oltra and Valero [24], and ordered metric frameworks with significant applications by Harandi and Emami [14].

In contrast to contractive mappings, expansive-type mappings arise in settings where distances may increase under iteration. A fundamental contribution in this direction was made by Wang [32], who introduced uniform expansive conditions of the form

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \geq \lambda\, d(\xi, \eta), \qquad \lambda > 1.$$

Subsequent studies established fixed point results for various classes of expansive mappings and their extensions; see Wang et al. [33], Park and Rhoades [27], and Taniguchi [31], as well as further developments in generalized metric spaces by Shahi [30] and Yeşilkaya and Aydın [35]. A common feature of these models is that the expansion mechanism depends solely on the interpoint distance $d(\xi, \eta)$ and does not involve the individual self–displacements $d(\xi, \mathscr{T}\xi)$ and $d(\eta, \mathscr{T}\eta)$. This limitation may restrict their applicability in situations where displacement effects play a crucial role.

On the other hand, displacement-sensitive contractive frameworks have proved to be highly effective in extending and unifying fixed point results. Notable examples include the α–ψ-contractive mappings introduced by Samet, Vetro and Vetro [29] and the F-contractions of Wardowski [34]. These approaches have been further developed in more general metric-type spaces by Jleli and Samet [17], [18], Pakhira [25], and Moussaoui [23], demonstrating that the incorporation of displacement terms provides enhanced flexibility and stronger analytical control.

Motivated by the absence of a systematic displacement-sensitive theory in the expansive setting, we introduce a rational displacement–distance gauge $\Psi_{\alpha,\beta}$ and define a new class of $\Psi_{\alpha,\beta}$–expansive mappings. This framework simultaneously incorporates interpoint distances and self–displacements into the expansion mechanism, thereby extending classical Wang-type conditions. In particular, the proposed class strictly enlarges the family of expansive mappings by including operators that do not satisfy any classical distance-based expansive condition.

The strength of the proposed generalization lies in its ability to capture a broader class of nonlinear operators through displacement-dependent control. The rational structure of the gauge $\Psi_{\alpha,\beta}$ allows adjustable sensitivity with respect to both distance and displacement, leading to greater flexibility in applications and improved convergence behavior under mild assumptions.

Recent developments further highlight the growing importance of fixed point methods in a wide range of applied problems, particularly in the study of integral equations and fractional differential equations. For instance, Panda et al. [26] employed fixed point techniques to establish existence results for multi-term fractional differential equations, demonstrating the effectiveness of such methods in analyzing complex nonlinear dynamics. Similarly, Alsaadi and Metwali [3] investigated coupled systems of Hammerstein–Urysohn integral equations, while Deep et al. [11] applied Darbo-type fixed point results to functional stochastic integral equations. Moreover, Metwali and Alsaadi [22] developed extensions of Darbo's fixed point theorem to treat systems of nonlinear integral equations, further illustrating the central role of fixed point theory in modern analysis and its applications.

In parallel, significant progress has been made in the development of generalized metric structures and advanced contraction frameworks, aimed at broadening the applicability of fixed point results. Jachymski [16] provided fundamental characterizations of generalized contractions, while Imdad et al. [15] extended these ideas to b-metric-type spaces with applications. Aydi et al. [4] investigated multi-dimensional fixed point results in generalized settings, and Alsaedi et al. [2] explored further applications in nonlinear analysis. More recently, Agarwal et al. [1] and Radenović et al. [28] have contributed to the advancement of fixed point theory in generalized and abstract metric frameworks, highlighting new directions and techniques.

These developments clearly indicate that incorporating additional structural features—such as generalized distances, nonlinear control functions, and displacement-dependent behavior—leads to more flexible and powerful analytical tools. This growing body of work strongly motivates the development of displacement-sensitive expansive frameworks, such as the one proposed in the present paper.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{R}^+ = [0, \infty)$. Let $\mathscr{X}$ be a nonempty set and let $d : \mathscr{X} \times \mathscr{X} \to \mathbb{R}^+$.

Definition 2.1. (Metric space [5, 12]) *The pair* $(\mathscr{X}, d)$ *is called a metric space if, for all* $\xi, \eta, \zeta \in \mathscr{X}$*, the following conditions hold:*

1. *$d(\xi, \eta) = 0$ if and only if $\xi = \eta$;*
2. *$d(\xi, \eta) = d(\eta, \xi)$;*
3. *$d(\xi, \zeta) \leq d(\xi, \eta) + d(\eta, \zeta)$.*

The classical Banach contraction principle [5] ensures that every contractive self-mapping on a complete metric space admits a unique fixed point, which can be obtained as the limit of successive iterations. This motivates the following standard notions.

Definition 2.2. (Fixed point, Picard iteration and convergence [5, 6, 12, 21]) *Let $(\mathscr{X}, d)$ be a metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a mapping.*

1. *A point $\pi \in \mathscr{X}$ is called a fixed point of $\mathscr{T}$ if $\mathscr{T}\pi = \pi$.*
2. *For any initial point $\xi_0 \in \mathscr{X}$, the sequence $\{\xi_n\}_{n\geq 0}$ defined by*

   $$\xi_{n+1} = \mathscr{T}\xi_n, \qquad n \geq 0,$$

   is called the Picard iterative sequence generated by $\mathscr{T}$ from ξ_0.
3. *The mapping $\mathscr{T}$ is called a Picard operator if it has a unique fixed point $\pi \in \mathscr{X}$ and, for every initial point $\xi_0 \in \mathscr{X}$, the corresponding Picard sequence converges to π, that is,*

   $$\lim_{n\to\infty} d(\xi_n, \pi) = 0.$$

Definition 2.3. (Convergence) *Let $(\mathscr{X}, d)$ be a metric space and let $\{\xi_n\}_{n\in\mathbb{N}} \subseteq \mathscr{X}$ be a sequence. We say that $\{\xi_n\}$ converges to $\pi \in \mathscr{X}$, and write $\xi_n \to \pi$, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$d(\xi_n, \pi) < \varepsilon \quad \textit{for all } n \geq N.$$

In this case, π is called the limit of the sequence.

Definition 2.4. (Cauchy sequence) *A sequence $\{\xi_n\}_{n\in\mathbb{N}} \subseteq \mathscr{X}$ is called a Cauchy sequence if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$d(\xi_m, \xi_n) < \varepsilon \quad \text{for all } m, n \geq N.$$

Definition 2.5. (Completeness) *A metric space $(\mathscr{X}, d)$ is said to be complete if every Cauchy sequence in $\mathscr{X}$ converges to a point in $\mathscr{X}$. Equivalently, for every sequence $\{\xi_n\}_{n\in\mathbb{N}} \subseteq \mathscr{X}$ there exists $\pi \in \mathscr{X}$ such that*

$$\lim_{n\to\infty} d(\xi_n, \pi) = 0.$$

These fundamental concepts will be used throughout the paper in the analysis of $\Psi_{\alpha,\beta}$–expansive mappings and their fixed point properties.

3. New Control Function

Classical expansive fixed point inequalities introduced by Wang and subsequently developed by several authors [27, 30, 31, 33, 35] describe expansion solely in terms of the interpoint distance $d(\xi, \eta)$. In such formulations, the individual self–displacements $d(\xi, \mathscr{T}\xi)$ and $d(\eta, \mathscr{T}\eta)$ do not influence the expansion mechanism. Consequently, these models may fail to capture finer structural features of nonlinear operators, particularly in situations where displacement effects play a significant role.

In contrast, displacement-sensitive contractive frameworks [29, 34] demonstrate that incorporating self-displacement terms leads to stronger and more flexible fixed point results. This observation motivates the development of a corresponding displacement-aware framework in the expansive setting.

To this end, we introduce a rational displacement–distance gauge that simultaneously accounts for the interpoint distance and the individual displacements.

Definition 3.1. (Rational displacement–distance gauge) *Let $\alpha, \beta > 0$. Define the function $\Psi_{\alpha,\beta} : [0, \infty)^3 \to [0, 1)$ by*

$$\Psi_{\alpha,\beta}(a, b, c) = \frac{c^\alpha}{1 + a^\beta + b^\beta + c^\alpha}, \qquad a, b, c \geq 0.$$

The function $\Psi_{\alpha,\beta}(a, b, c)$ measures the relative influence of the interpoint distance c compared to the displacement magnitudes a and b. When the displacements a and b are large, the denominator increases and the value of the gauge decreases, resulting in a weaker expansion effect. Conversely, when the interpoint distance c is dominant, the gauge increases and strengthens the expansion mechanism. The parameters α and β control the sensitivity of the gauge with respect to distance and displacement, respectively.

Proposition 3.2. (Basic properties) *For every $\alpha, \beta > 0$ and all $a, b, c \geq 0$, the function $\Psi_{\alpha,\beta}$ satisfies:*

1. $0 \leq \Psi_{\alpha,\beta}(a, b, c) < 1$ *and* $\Psi_{\alpha,\beta}(a, b, 0) = 0$;
2. $\Psi_{\alpha,\beta}$ *is strictly increasing in c on $(0, \infty)$ and strictly decreasing in each of a and b on $(0, \infty)$; these properties extend to $[0, \infty)$ by continuity;*
3. *if there exists $\theta \geq 0$ such that $a^\beta + b^\beta \leq \theta c^\alpha$, then*

$$\Psi_{\alpha,\beta}(a, b, c) \geq \frac{c^\alpha}{1 + (1 + \theta)c^\alpha}.$$

Proof. Let $\alpha, \beta > 0$ and $a, b, c \geq 0$.

(1) Bounds. Since $c^\alpha \geq 0$ and

$$1 + a^\beta + b^\beta + c^\alpha > 0,$$

we have $\Psi_{\alpha,\beta}(a, b, c) \geq 0$. If $c = 0$, then $\Psi_{\alpha,\beta}(a, b, 0) = 0$. If $c > 0$, then

$$\Psi_{\alpha,\beta}(a, b, c) < \frac{c^\alpha}{c^\alpha} = 1,$$

which yields $0 \leq \Psi_{\alpha,\beta}(a, b, c) < 1$.

(2) Monotonicity. For fixed $a, b \geq 0$ and $c > 0$, differentiation gives

$$\frac{\partial}{\partial c}\Psi_{\alpha,\beta}(a, b, c) = \frac{\alpha c^{\alpha-1}(1 + a^\beta + b^\beta)}{(1 + a^\beta + b^\beta + c^\alpha)^2} > 0,$$

so $\Psi_{\alpha,\beta}$ is strictly increasing in c.

For fixed $b, c \geq 0$ and $a > 0$,

$$\frac{\partial}{\partial a}\Psi_{\alpha,\beta}(a, b, c) = -\frac{\beta a^{\beta-1} c^\alpha}{(1 + a^\beta + b^\beta + c^\alpha)^2} < 0 \quad \text{whenever } c > 0,$$

showing strict decrease in a. The same argument applies to b. Continuity of $\Psi_{\alpha,\beta}$ extends these properties to the boundary points.

(3) Lower bound under domination. Assume that $a^\beta + b^\beta \leq \theta c^\alpha$. Then

$$1 + a^\beta + b^\beta + c^\alpha \leq 1 + (1 + \theta)c^\alpha.$$

Since all quantities are positive, taking reciprocals yields

$$\frac{1}{1 + a^\beta + b^\beta + c^\alpha} \geq \frac{1}{1 + (1 + \theta)c^\alpha}.$$

Multiplying by $c^\alpha \geq 0$ gives

$$\Psi_{\alpha,\beta}(a, b, c) \geq \frac{c^\alpha}{1 + (1 + \theta)c^\alpha},$$

which completes the proof.

4. $\Psi_{\alpha,\beta}$–Expansive Mappings

Classical expansive mappings introduced by Wang and further developed by several authors are characterized by inequalities in which the expansion factor depends solely on the interpoint distance $d(\xi, \eta)$. Such conditions do not incorporate the individual self–displacements $d(\xi, \mathscr{T}\xi)$ and $d(\eta, \mathscr{T}\eta)$, and therefore may fail to capture finer structural properties of nonlinear operators, especially in situations where displacement effects are significant.

To overcome this limitation, we employ the rational displacement–distance gauge $\Psi_{\alpha,\beta}$ introduced in the previous section and define a new class of expansive mappings.

Definition 4.1. ($\Psi_{\alpha,\beta}$–expansive mapping) *Let $(\mathscr{X}, d)$ be a metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a mapping. We say that $\mathscr{T}$ is $\Psi_{\alpha,\beta}$–expansive if there exists a constant $\lambda > 0$ such that, for all $\xi, \eta \in \mathscr{X}$ with $\xi \neq \eta$,*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \geq \Big(1 + \lambda\, \Psi_{\alpha,\beta}\big(d(\xi, \mathscr{T}\xi),\, d(\eta, \mathscr{T}\eta),\, d(\xi, \eta)\big)\Big)\, d(\xi, \eta).$$

The above inequality defines a nonlinear expansive mechanism in which the strength of expansion is modulated by displacement terms. When the self–displacements are small relative to the interpoint distance, the value of $\Psi_{\alpha,\beta}$ becomes large, resulting in stronger expansion. Conversely, large displacements reduce the value of the gauge and weaken the expansion, thereby allowing more flexible behavior beyond classical distance-based conditions.

Remark 4.2. *If*

$$d(\xi, \mathscr{T}\xi) = d(\eta, \mathscr{T}\eta) = 0,$$

then

$$\Psi_{\alpha,\beta}(0, 0, d(\xi, \eta)) = \frac{d(\xi, \eta)^\alpha}{1 + d(\xi, \eta)^\alpha},$$

and Definition 4.1 reduces to a rational expansive condition depending only on $d(\xi, \eta)$. Thus, the class of $\Psi_{\alpha,\beta}$–expansive mappings naturally extends classical distance-based expansive models.

Remark 4.3. *A classical Wang-type expansive mapping satisfies the inequality*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \geq \lambda\, d(\xi, \eta), \qquad \lambda > 1,$$

for all $\xi, \eta \in \mathscr{X}$. In this formulation, the expansion factor λ is constant and depends only on the interpoint distance. In particular, the behavior of the mapping at individual points, measured by the self–displacements $d(\xi, \mathscr{T}\xi)$ and $d(\eta, \mathscr{T}\eta)$, does not influence the inequality.

In contrast, a $\Psi_{\alpha,\beta}$–expansive mapping satisfies

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \geq \Big(1 + \lambda\, \Psi_{\alpha,\beta}\big(d(\xi, \mathscr{T}\xi),\, d(\eta, \mathscr{T}\eta),\, d(\xi, \eta)\big)\Big) d(\xi, \eta),$$

where the expansion factor

$$1 + \lambda\, \Psi_{\alpha,\beta}(a, b, c) = 1 + \lambda\, \frac{c^{\alpha}}{1 + a^{\beta} + b^{\beta} + c^{\alpha}}$$

depends simultaneously on the interpoint distance $c = d(\xi, \eta)$ *and the self–displacements* $a = d(\xi, \mathscr{T}\xi)$ *and* $b = d(\eta, \mathscr{T}\eta)$.

This leads to the following fundamental differences:

1. *In the classical case, the expansion factor* λ *is fixed, whereas in the* $\Psi_{\alpha,\beta}$ *framework the effective factor*

   $$1 + \lambda\, \Psi_{\alpha,\beta}(a, b, c)$$

   varies with the points. Thus, the expansion adapts to the local behavior of the mapping.

2. *Classical expansive mappings ignore the quantities* $d(\xi, \mathscr{T}\xi)$ *and* $d(\eta, \mathscr{T}\eta)$. *In contrast, these terms directly influence* $\Psi_{\alpha,\beta}$. *Large displacements increase the denominator and reduce the expansion, while small displacements enhance it. This yields a finer and more responsive control mechanism.*

3. *If the displacements vanish, that is,*

   $$d(\xi, \mathscr{T}\xi) = d(\eta, \mathscr{T}\eta) = 0,$$

   then

   $$\Psi_{\alpha,\beta}(0, 0, c) = \frac{c^{\alpha}}{1 + c^{\alpha}},$$

   and the inequality reduces to a distance-dependent expansive condition. Hence, classical-type behavior is recovered as a particular case.

4. *There exist mappings for which the classical inequality*

   $$d(\mathscr{T}\xi, \mathscr{T}\eta) \geq \lambda d(\xi, \eta)$$

   fails for every $\lambda > 1$, *while the* $\Psi_{\alpha,\beta}$ *condition remains valid due to the compensating effect of displacement terms. This demonstrates that the proposed framework applies to a strictly larger class of nonlinear operators.*

5. *The parameters* α *and* β *allow independent adjustment of the sensitivity with respect to distance and displacement. This flexibility is absent in classical Wang-type models and is advantageous in applications.*

Therefore, $\Psi_{\alpha,\beta}$–expansive mappings provide a genuine generalization of classical expansive mappings by replacing a rigid constant expansion factor with a displacement-sensitive and adaptive mechanism.

5. Main Result

For expansive mappings, the classical forward Picard iteration $\xi_{n+1} = \mathscr{T}\xi_n$ is not, in general, an appropriate convergence scheme, since distances may increase under iteration. In such situations, a natural alternative is the backward Picard iteration (also called a backward orbit): given an initial point $\xi_0 \in \mathscr{X}$, one constructs a sequence $\{\xi_n\}_{n\geq 0}$ satisfying

$$\mathscr{T}\xi_{n+1} = \xi_n, \qquad n \geq 0.$$

The existence of such a sequence requires that $\mathscr{T}$ be surjective.

Theorem 5.1. *Let $(\mathscr{X}, d)$ be a complete metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a surjective $\Psi_{\alpha,\beta}$–expansive mapping. Assume that there exists a constant $\theta \geq 0$ such that, for all $\xi, \eta \in \mathscr{X}$ with $\xi \neq \eta$,*

$$d(\xi, \mathscr{T}\xi)^{\beta} + d(\eta, \mathscr{T}\eta)^{\beta} \leq \theta\, d(\xi, \eta)^{\alpha}. \tag{1}$$

Then $\mathscr{T}$ admits a unique fixed point $\pi \in \mathscr{X}$. Moreover, for each initial point $\xi_0 \in \mathscr{X}$, every backward Picard sequence $\{\xi_n\}_{n\geq 0}$ satisfying $\mathscr{T}\xi_{n+1} = \xi_n$ converges to π.

Proof. Let $\xi, \eta \in \mathscr{X}$ with $\xi \neq \eta$ and define

$$a = d(\xi, \mathscr{T}\xi), \quad b = d(\eta, \mathscr{T}\eta), \quad c = d(\xi, \eta).$$

By (1) and Proposition 3.2, we obtain

$$\Psi_{\alpha,\beta}(a, b, c) \geq \frac{c^{\alpha}}{1 + (1+\theta)c^{\alpha}}.$$

Substituting into the $\Psi_{\alpha,\beta}$–expansive inequality yields

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \geq \left(1 + \lambda \frac{c^{\alpha}}{1 + (1+\theta)c^{\alpha}}\right) c.$$

Rearranging, we obtain

$$c \leq q(c)\, d(\mathscr{T}\xi, \mathscr{T}\eta), \qquad q(c) = \frac{1 + (1+\theta)c^{\alpha}}{1 + (1+\theta+\lambda)c^{\alpha}}, \tag{2}$$

where $0 < q(c) < 1$ for all $c > 0$.
Let $\xi_0 \in \mathscr{X}$. Since $\mathscr{T}$ is surjective, there exists a sequence $\{\xi_n\}_{n\geq 0}$ such that $\mathscr{T}\xi_{n+1} = \xi_n$. Define

$$c_n = d(\xi_{n+1}, \xi_n).$$

Applying (2) to (ξ_{n+1}, ξ_n), we obtain

$$c_n \leq q(c_n)c_{n-1}.$$

Thus $\{c_n\}$ is decreasing and bounded below by 0, hence $c_n \to \ell \geq 0$.

Assume $\ell > 0$. Passing to the limit and using continuity of q, we obtain

$$\ell \leq q(\ell)\,\ell,$$

which is impossible since $0 < q(\ell) < 1$. Therefore $\ell = 0$, i.e.,

$$d(\xi_{n+1}, \xi_n) \to 0. \tag{3}$$

We now prove that $\{\xi_n\}$ is a Cauchy sequence. Since $q(c) \to 1$ as $c \to 0$, there exist $q_0 \in (0, 1)$ and $N \in \mathbb{N}$ such that $q(c_n) \leq q_0$ for all $n \geq N$. Hence, for all $n \geq N$,

$$c_n \leq q_0\, c_{n-1}.$$

By induction,

$$c_n \leq q_0^{n-N} c_N.$$

Thus the series $\sum_{k\geq 0} c_k$ converges by comparison with a geometric series, and consequently $\{\xi_n\}$ is a Cauchy sequence.

Since $(\mathscr{X}, d)$ is complete, there exists $\pi \in \mathscr{X}$ such that $\xi_n \to \pi$.
We now show that π is a fixed point. Using $\mathscr{T}\xi_{n+1} = \xi_n$, we have

$$d(\mathscr{T}\pi, \pi) \leq d(\mathscr{T}\pi, \mathscr{T}\xi_{n+1}) + d(\xi_n, \pi).$$

From (2), applied to (π, ξ_{n+1}), we obtain

$$d(\pi, \xi_{n+1}) \leq q(d(\pi, \xi_{n+1}))\, d(\mathscr{T}\pi, \mathscr{T}\xi_{n+1}),$$

which implies

$$d(\mathscr{T}\pi, \mathscr{T}\xi_{n+1}) \leq \frac{1}{q(d(\pi, \xi_{n+1}))} d(\pi, \xi_{n+1}).$$

Since $d(\pi, \xi_{n+1}) \to 0$ and $q(t) \to 1$ as $t \to 0$, the right-hand side tends to 0, and hence $d(\mathscr{T}\pi, \mathscr{T}\xi_{n+1}) \to 0$. Since $\mathscr{T}\xi_{n+1} = \xi_n \to \pi$, it follows that

$$d(\mathscr{T}\pi, \pi) = 0,$$

and therefore $\mathscr{T}\pi = \pi$.
Finally, let π' be another fixed point. Applying (2) to (π, π'), we obtain

$$d(\pi, \pi') \leq q(d(\pi, \pi'))\, d(\pi, \pi').$$

Since $0 < q(\cdot) < 1$, this implies $d(\pi, \pi') = 0$, hence $\pi = \pi'$.

Remark 5.2. *The domination condition (1) is an additional structural assumption and does not follow automatically from the $\Psi_{\alpha,\beta}$–expansive property. Indeed, the definition of $\Psi_{\alpha,\beta}$–expansiveness allows the displacement terms $d(\xi, \mathscr{T}\xi)$ and $d(\eta, \mathscr{T}\eta)$ to behave independently of the interpoint distance $d(\xi, \eta)$.*

The role of condition (1) is to ensure that these displacements remain sufficiently controlled relative to $d(\xi, \eta)$. More precisely, it prevents the denominator in the gauge

$$\Psi_{\alpha,\beta}(a, b, c) = \frac{c^{\alpha}}{1 + a^{\beta} + b^{\beta} + c^{\alpha}}$$

from being dominated by the displacement terms $a^{\beta} + b^{\beta}$. Without such control, the value of $\Psi_{\alpha,\beta}$ may become arbitrarily small even when $c = d(\xi, \eta)$ is large, thereby weakening the expansion mechanism and obstructing convergence.

From an analytical viewpoint, condition (1) can be interpreted as a growth restriction on the mapping $\mathscr{T}$, ensuring that the displacement behaves at most polynomially with respect to the interpoint distance. Such balance conditions are common in nonlinear analysis and play a crucial role in establishing convergence results.

For instance, mappings with polynomial-type growth (such as those presented in Section 6) typically satisfy (1), whereas mappings with rapidly increasing displacement (e.g., exponential-type growth) may fail to satisfy it, even if they satisfy a $\Psi_{\alpha,\beta}$–expansive inequality. Therefore, the domination condition is natural but nontrivial and is essential for ensuring the effectiveness of the gauge and the convergence of backward Picard iterations.

Remark 5.3. *Every classical Wang-type expansive mapping satisfies*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \geq \lambda_0\, d(\xi, \eta), \qquad \lambda_0 > 1,$$

and is therefore $\Psi_{\alpha,\beta}$–expansive for a suitable choice of $\lambda > 0$. Indeed, since

$$0 \leq \Psi_{\alpha,\beta}(a, b, c) < 1,$$

we have

$$1 + \lambda\, \Psi_{\alpha,\beta}(a, b, c) \leq 1 + \lambda = \lambda_0,$$

so the $\Psi_{\alpha,\beta}$–expansive inequality is weaker than the classical Wang condition.

However, the converse does not hold in general. In the $\Psi_{\alpha,\beta}$ framework, the effective expansion factor

$$1 + \lambda\,\Psi_{\alpha,\beta}(a,b,c) = 1 + \lambda\,\frac{c^{\alpha}}{1 + a^{\beta} + b^{\beta} + c^{\alpha}}$$

depends explicitly on the displacement terms and may vary from point to point. In particular, this factor can be strictly smaller than any fixed constant $\lambda_0 > 1$, even though the $\Psi_{\alpha,\beta}$–expansive inequality remains valid.

This variability allows the proposed framework to capture mappings that exhibit non-uniform or locally adaptive expansive behavior, which cannot be treated within the classical Wang-type setting. Consequently, the class of $\Psi_{\alpha,\beta}$–expansive mappings strictly contains the class of Wang-type expansive mappings.

This strict inclusion is not merely formal but has concrete implications: there exist mappings (see Section 6) that satisfy the $\Psi_{\alpha,\beta}$–expansive condition but fail to satisfy any uniform Wang-type inequality. Therefore, Theorem 5.1 provides a genuine and nontrivial extension of the classical Wang fixed point principle.

6. Examples

This section provides representative examples that illustrate the scope, effectiveness, and strict generality of the proposed $\Psi_{\alpha,\beta}$–expansive framework. The aim is twofold: first, to verify the applicability of the main theorem by explicitly constructing mappings that satisfy all the required conditions; and second, to demonstrate that the present approach genuinely extends the classical theory of expansive mappings.

In particular, we exhibit mappings that fail to satisfy any uniform Wang-type expansive condition, yet still fulfill the $\Psi_{\alpha,\beta}$–expansive inequality due to the presence of displacement-dependent control. This confirms that the incorporation of self–displacement terms leads to a strictly larger and more flexible class of operators.

Furthermore, the examples highlight how the domination condition is naturally satisfied in typical nonlinear settings, while also clarifying its role in ensuring convergence of backward Picard iterations.

Example 6.1. Let $(\mathscr{X}, d) = (\mathbb{R}, |\cdot|)$ and define

$$\mathscr{T}(\xi) = \xi + \xi^3, \qquad \xi \in \mathbb{R}.$$

Then $\mathscr{T}$ is strictly increasing and satisfies $\lim_{\xi\to\pm\infty} \mathscr{T}(\xi) = \pm\infty$, hence it is bijective. The fixed point equation

$$\mathscr{T}(\pi) = \pi \quad \Longleftrightarrow \quad \pi + \pi^3 = \pi$$

yields $\pi^3 = 0$, and thus the unique fixed point is $\pi = 0$.

For $\xi \neq \eta$, we compute

$$|\mathscr{T}\xi - \mathscr{T}\eta| = |(\xi - \eta) + (\xi^3 - \eta^3)| = |\xi - \eta|\big(1 + \xi^2 + \xi\eta + \eta^2\big).$$

Using the identity

$$\xi^2 + \xi\eta + \eta^2 = \tfrac{1}{4}(\xi - \eta)^2 + \tfrac{3}{4}(\xi + \eta)^2 \geq \tfrac{1}{4}(\xi - \eta)^2,$$

we obtain

$$|\mathscr{T}\xi - \mathscr{T}\eta| \geq \Big(1 + \tfrac{1}{4}|\xi - \eta|^2\Big)|\xi - \eta|.$$

Fix $\alpha = 2$ and $\beta = 1$, and set

$$a = |\xi^3|, \qquad b = |\eta^3|, \qquad c = |\xi - \eta|.$$

Then

$$\Psi_{2,1}(a, b, c) = \frac{c^2}{1 + a + b + c^2} \leq c^2.$$

Multiplying by $\lambda = \frac{1}{4}$, we obtain

$$\lambda\,\Psi_{2,1}(a, b, c) \leq \tfrac{1}{4}c^2, \quad \text{and hence} \quad 1 + \lambda\,\Psi_{2,1}(a, b, c) \leq 1 + \tfrac{1}{4}c^2.$$

On the other hand, from the previous estimate we have

$$|\mathscr{T}\xi - \mathscr{T}\eta| \geq \Big(1 + \tfrac{1}{4}c^2\Big)c.$$

Therefore, combining the above inequalities yields

$$(1 + \lambda\,\Psi_{2,1}(a, b, c))c \leq \Big(1 + \tfrac{1}{4}c^2\Big)c \leq |\mathscr{T}\xi - \mathscr{T}\eta|.$$

Hence,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \geq \Big(1 + \lambda\,\Psi_{2,1}(a, b, c)\Big)d(\xi, \eta),$$

which proves that $\mathscr{T}$ is $\Psi_{2,1}$–expansive.

Finally, consider the backward orbit $\{\xi_n\}$ defined by $\xi_{n+1} = \mathscr{T}^{-1}(\xi_n)$. Since $\mathscr{T}$ is strictly increasing and satisfies $\mathscr{T}(\xi) > \xi$ for $\xi > 0$ and $\mathscr{T}(\xi) < \xi$ for $\xi < 0$, it follows that

$$|\xi_{n+1}| < |\xi_n| \quad \text{whenever } \xi_n \neq 0.$$

Thus, $\{|\xi_n|\}$ is decreasing and bounded below by 0, hence convergent. Let $\ell = \lim_{n\to\infty} |\xi_n|$. Passing to the limit in

$$\xi_n = \mathscr{T}(\xi_{n+1}) = \xi_{n+1} + \xi_{n+1}^3,$$

we obtain $\ell = \ell + \ell^3$, which implies $\ell = 0$. Therefore, $\xi_n \to 0 = \pi$.
Moreover, it is straightforward to verify that the domination condition (1) is satisfied for this mapping; hence, by Theorem 5.1, $\mathscr{T}$ admits a unique fixed point and every backward Picard sequence converges to $\pi = 0$.

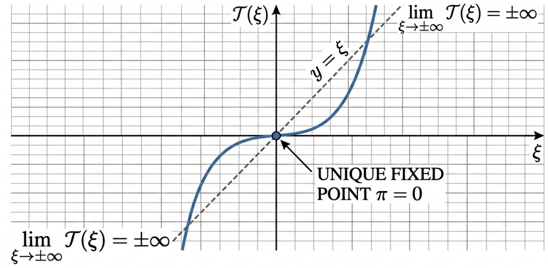

Figure 1: The mapping $\mathscr{T}(\xi) = \xi + \xi^3$

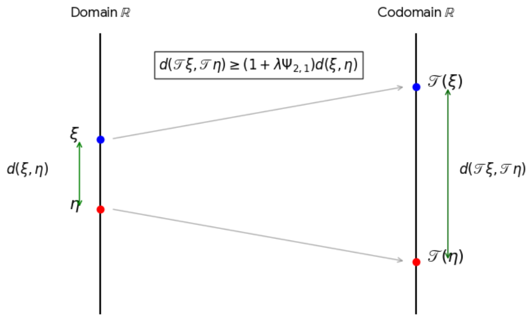

Figure 2: $\Psi_{2,1}$-expansive condition

Example 6.2. (Strict inclusion of $\Psi_{\alpha,\beta}$–expansive class). Let $(\mathscr{X}, d) = ([0,\infty), |\cdot|)$ and define

$$\mathscr{T}(\xi) = \xi + \xi^2, \qquad \xi \geq 0.$$

The mapping $\mathscr{T}$ is continuous and strictly increasing, since $\mathscr{T}'(\xi) = 1 + 2\xi > 0$ for all $\xi \geq 0$. Moreover, $\mathscr{T}(0) = 0$ and $\lim_{\xi\to\infty} \mathscr{T}(\xi) = \infty$, so $\mathscr{T}$ is bijective. The fixed point equation

$$\mathscr{T}(\pi) = \pi \quad \Longleftrightarrow \quad \pi + \pi^2 = \pi$$

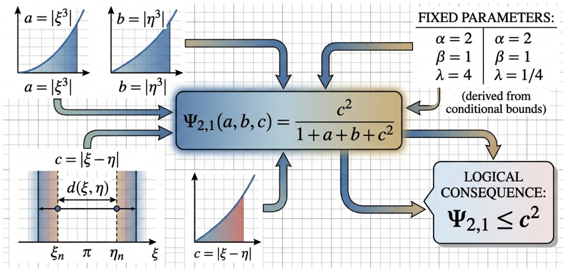

Figure 3: Deconstructing $\Psi_{2,1}$ parameters

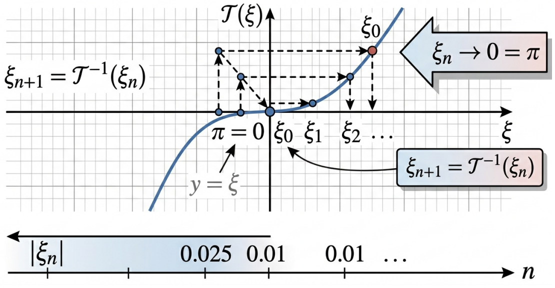

Figure 4: Backward orbit $\{\xi_n\}$convergence

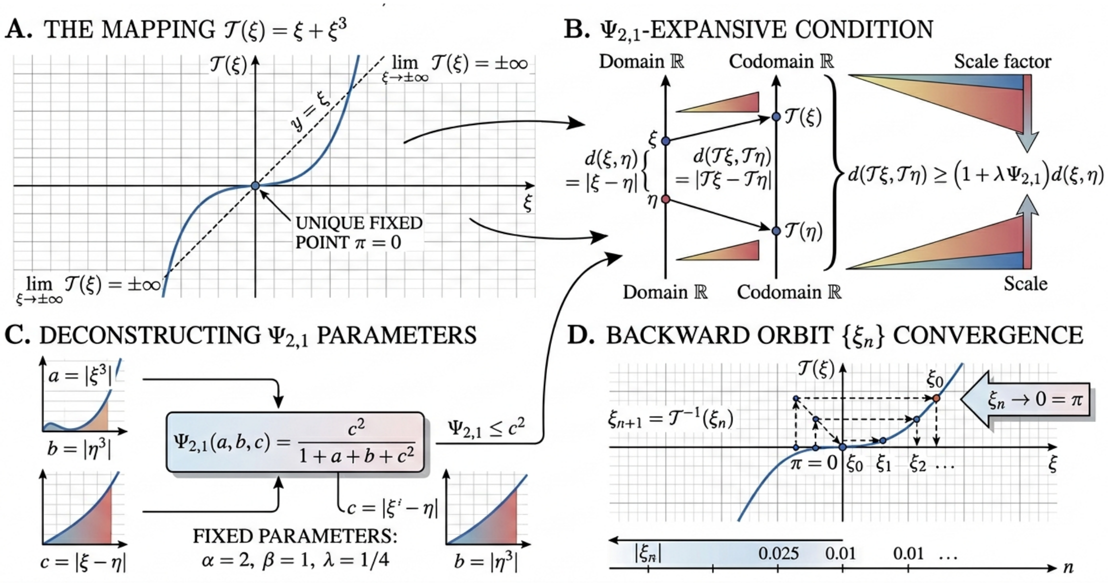

Figure 5: Visualization of $\Psi_{2,1}$-expansive mapping and backward orbit

yields the unique fixed point $\pi = 0$.

Failure of Wang–type expansiveness. For $\xi \neq \eta$,

$$|\mathscr{T}\xi - \mathscr{T}\eta| = |\xi - \eta|(1 + \xi + \eta),$$

so that

$$\frac{|\mathscr{T}\xi - \mathscr{T}\eta|}{|\xi - \eta|} = 1 + \xi + \eta.$$

Since $\inf_{\xi,\eta\geq 0}(1 + \xi + \eta) = 1$, there is no constant $\lambda_0 > 1$ such that

$$|\mathscr{T}\xi - \mathscr{T}\eta| \geq \lambda_0|\xi - \eta| \quad \text{for all } \xi \neq \eta.$$

Hence, $\mathscr{T}$ is not Wang–type expansive.

$\Psi_{1,1}$–expansiveness. Fix $\alpha = \beta = 1$ and set

$$a = \xi^2, \quad b = \eta^2, \quad c = |\xi - \eta|.$$

Then

$$\Psi_{1,1}(a, b, c) = \frac{c}{1 + \xi^2 + \eta^2 + c} \leq \frac{c}{1 + c} \leq c.$$

Thus, for any $\lambda \in (0, 1]$,

$$\lambda\,\Psi_{1,1}(a, b, c) \leq c.$$

Since $c \leq \xi + \eta$ for all $\xi, \eta \geq 0$, it follows that

$$1 + \lambda\,\Psi_{1,1}(a, b, c) \leq 1 + \xi + \eta.$$

Consequently,

$$(1 + \lambda\,\Psi_{1,1}(a, b, c))c \leq (1 + \xi + \eta)c = |\mathscr{T}\xi - \mathscr{T}\eta|.$$

Therefore,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \geq \Big(1 + \lambda\,\Psi_{1,1}(a, b, c)\Big)d(\xi, \eta),$$

which shows that $\mathscr{T}$ is $\Psi_{1,1}$–expansive.

The mapping $\mathscr{T}$ is $\Psi_{\alpha,\beta}$–expansive but not Wang–type expansive. This proves that the class of $\Psi_{\alpha,\beta}$–expansive mappings strictly extends the classical Wang-type class.

7. Conclusion

In this paper, we have developed a new displacement–sensitive expansive framework based on the rational gauge $\Psi_{\alpha,\beta}$, which simultaneously incorporates interpoint distances and individual self–displacements. This approach provides a significant extension of classical expansive models, including Wang–type inequalities, where the expansion mechanism depends solely on the interpoint distance.

Under a natural domination condition linking displacement to distance, we established the existence, uniqueness, and global convergence of fixed points for $\Psi_{\alpha,\beta}$–expansive mappings in complete metric spaces. The use of backward Picard iteration offers an appropriate and effective convergence scheme in the expansive setting, where forward iteration may fail.

The examples presented in this work demonstrate that the proposed class strictly contains mappings that are not Wang–type expansive and cannot be treated within existing distance-based frameworks. This confirms that the incorporation of displacement into the expansive condition leads to a substantially broader and more flexible theory.

Furthermore, the displacement–sensitive nature of the proposed framework suggests potential applications in areas where expansion depends on the state of the system, such as nonlinear dynamical systems, fractional and ordinary differential equations, and integral equations with non-uniform growth. In such contexts, the ability to incorporate local displacement information may provide improved tools for modeling, stability analysis, and convergence behavior.

Future research may focus on extending this theory to generalized metric structures, including b–metric spaces, partial metric spaces, and perturbed metric settings, as well as exploring further applications to nonlinear differential, integral, and discrete dynamical systems.

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